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NONLINEAR EVOLUTION EQUATIONS AND PRODUCT INTEGRATION IN BANACH SPACES

BY
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Abstract. The method of product integration is used to obtain solutions to the nonlinear evolution equation $g'=Ag$ where A is a function from a Banach space S to itself and g is a continuously differentiable function from $[0, \infty)$ to S. The conditions required on A are that A is dissipative on S, the range of $(e-\varepsilon A)=S$ for all $\varepsilon \geqq 0$, and A is continuous on S.

1. **Introduction.** Let S be a Banach space and let A be a mapping from a subset of S to S. An evolution equation is a system $g'=A(g)$, $g(0)=p$, where g is a continuous function from $[0, \infty)$ to S and p is a point in S. In [3] F. Browder has considered nonlinear evolution equations in which S is the Hilbert space and A is continuous, bounded, and dissipative on S. In recent articles Y. Kōmura [12], T. Kato [10], and M. Crandall and A. Pazy [5] have considered nonlinear evolution equations in which S is the Hilbert space and A is maximal dissipative, not necessarily continuous, and is the infinitesimal generator of a semigroup of nonlinear nonexpansive transformations on S.

The object of this paper is to obtain solutions to an evolution system in a general Banach space using the method of product integration. A definition of product integration is given as follows:

Suppose that p is in S, $x>0$, and z is a point in S such that if $c>0$ there exists a chain $\{s_i\}_{i=0}^m$ from 0 to x such that if $\{t_i\}_{i=0}^n$ is a refinement of $\{s_i\}_{i=0}^m$ then

$$(1) \qquad \left\| z-\prod_{i=1}^{n} (e-(t_i-t_{i-1})A)^{-1}p \right\| < c.$$

(Note that e denotes the identity map on S, $(e-(t_i-t_{i-1})A)^{-1}$ denotes the inverse map of $(e-(t_i-t_{i-1})A)$, $\prod_{i=1}^{1} (e-(t_i-t_{i-1})A)^{-1}p=(e-(t_1-t_0)A)^{-1}p$, and if j is an integer in $[2, n]$

$$\prod_{i=1}^{j} (e-(t_i-t_{i-1})A)^{-1}p = (e-(t_j-t_{j-1})A)^{-1} \prod_{i=1}^{j-1} (e-(t_i-t_{i-1})A)^{-1}p,$$

where the product operation is composition of mappings.) Then z is said to be the product integral of A with respect to p from 0 to x and is denoted by $\prod_0^x(e-dIA)^{-1}p$.

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In [1] G. Birkhoff and in [16] J. Neuberger have used product integration to solve evolution systems where the mapping A is Lipschitz continuous. In this paper the product integration method will be extended to mappings not necessarily Lipschitz continuous.

2. **An existence theorem.** Let A be a mapping from a subset of S to S such that the following are true:

(I) A is dissipative on its domain D_A, i.e., if $u, v \in D_A$ and $\varepsilon \geqq 0$ then $\|(e-\varepsilon A)u-(e-\varepsilon A)v\| \geqq \|u-v\|$.

(II) There is an open subset C of S such that $C \subseteq D_A$ and a positive number α such that if $0 \leqq \varepsilon < \alpha$ then $C \subseteq R_{(e-\varepsilon A)}$ (where $R_{(e-\varepsilon A)}$ denotes the range of $(e-\varepsilon A)$).

(III) A is continuous on C.

Note that by (I) if $\varepsilon > 0$ then $(e-\varepsilon A)$ is 1-1 on D_A and by (II) if $0 \leqq \varepsilon < \alpha$ and $q \in C$ then $q \in D_{(e-\varepsilon A)^{-1}} = R_{(e-\varepsilon A)}$. If $0 \leqq \varepsilon < \alpha$ and $q \in R_{(e-\varepsilon A)}$ let $L(\varepsilon)q = (e-\varepsilon A)^{-1}q$. By (I) $L(\varepsilon)$ is nonexpansive on $R_{(e-\varepsilon A)}$, i.e., if $u, v \in R_{(e-\varepsilon A)}$ then

$$\|L(\varepsilon)u - L(\varepsilon)v\| \leqq \|u-v\|. \tag{2}$$

THEOREM. *Let A satisfy conditions* (I), (II), *and* (III). *If $p \in C$ and*

$$\gamma_p = \min\{\operatorname{dist}(p, \partial C)/\|Ap\|, \alpha\},$$

then there is a continuously differentiable function g_p from $[0, \gamma_p)$ to S such that $g_p(0) = p$ and if $0 \leqq x < \gamma_p$, $g_p'(x) = Ag_p(x)$ and $g_p(x) = \prod_0^x (e-dIA)^{-1}p$.

The theorem will be proved by means of a sequence of lemmas each of which is under the hypothesis of the theorem.

LEMMA 1.1. *If $q \in C$ and $0 \leqq x, y < \alpha$, then $\|L(x)q - L(y)q\| \leqq |x-y| \cdot \|Aq\|$.*

Proof. Using (2) we have that

$$\begin{aligned}
\|L(x)q - L(y)q\| &= \|L(x)q - L(x)(e-xA)L(y)q\| \\
&\leqq \|q-(e-xA)L(y)q\| \\
&= \|q - [(x/y)(e-yA)L(y)q + (1-x/y)L(y)q]\| \\
&= |1-x/y|\ \|q-L(y)q\| \\
&\leqq |1-x/y|\ \|(e-yA)q-q\| \\
&= |x-y|\ \|Aq\|.
\end{aligned}$$

LEMMA 1.2. *Let $q \in C$, let $0 < x < \gamma_q$, and let $\{s_i\}_{i=0}^m$ be a chain from 0 to x. If j is an integer in $[1, m]$ then*

$$\prod_{i=1}^{j-1} L(s_i - s_{i-1})q \in C, \tag{3}$$

$$\left\| \prod_{i=1}^{j} L(s_i - s_{i-1})q - q \right\| \leqq s_j \|Aq\|, \tag{4}$$

and

$$\left\| A \prod_{i=1}^{j} L(s_i - s_{i-1})q \right\| \leqq \|Aq\|. \tag{5}$$

(*Note that* $\prod_{i=1}^{0} L(s_i - s_{i-1})$ *denotes the identity map, i.e.,* $\prod_{i=1}^{0} L(s_i - s_{i-1})q = q$.)

Proof. The proof is by induction. For $j=1$ $\prod_{i=1}^{j-1} L(s_i - s_{i-1})q = q \in C$,

$$\left\| \prod_{i=1}^{1} L(s_i - s_{i-1})q - q \right\| \leqq s_1 \cdot \|Aq\|$$

(by Lemma 1.1), and

$$\left\| A \prod_{i=1}^{1} L(s_i - s_{i-1})q \right\| = \|1/s_1[L(s_1 - s_0)q - q]\| \leqq \|Aq\|.$$

Suppose that j is an integer in $[1, m-1]$, $\prod_{i=1}^{j-1} L(s_i - s_{i-1})q \in C$,

$$\left\| \prod_{i=1}^{j} L(s_i - s_{i-1})q - q \right\| \leqq s_j \cdot \|Aq\|,$$

and $\|A \prod_{i=1}^{j} L(s_i - s_{i-1})q\| \leqq \|Aq\|$. Then,

$$\prod_{i=1}^{j} L(s_i - s_{i-1})q \in C \subseteq D_{L(s_{j+1} - s_j)}.$$

Further,

$$\left\| \prod_{i=1}^{j+1} L(s_i - s_{i-1})q - q \right\| = \left\| \sum_{i=1}^{j+1} \left[\prod_{k=i}^{j+1} L(s_k - s_{k-1})q - \prod_{k=i+1}^{j+1} L(s_k - s_{k-1})q \right] \right\|$$

(note that $\prod_{k=j+2}^{j+1} L(s_k - s_{k-1})$ is the identity map)

$$\begin{aligned} &\leqq \sum_{i=1}^{j+1} \|L(s_i - s_{i-1})q - q\| \\ &\leqq s_{j+1} \cdot \|Aq\|. \end{aligned}$$

Moreover,

$$\begin{aligned} \left\| A \prod_{i=1}^{j+1} L(s_i - s_{i-1})q \right\| &= \left\| \left(\frac{1}{s_{j+1} - s_j} \right) \left[\prod_{i=1}^{j+1} L(s_i - s_{i-1})q - \prod_{i=1}^{j} L(s_i - s_{i-1})q \right] \right\| \\ &\leqq \left\| A \prod_{i=1}^{j} L(s_i - s_{i-1})q \right\| \\ &\leqq \|Aq\|. \end{aligned}$$

LEMMA 1.3. *Let* $q \in C$, *let* $0 < x < \gamma_q$, *and let* $\{t_i\}_{i=0}^{n}$ *be a chain from* 0 *to* x. *If* j *is an integer in* $[1, n]$ *then*

$$\prod_{i=j}^{n} L(t_i - t_{i-1})q - q = \sum_{i=j}^{n} (t_i - t_{i-1}) A \prod_{k=j}^{i} L(t_k - t_{k-1})q. \tag{6}$$

Proof.

$$\prod_{i=j}^{n} L(t_i-t_{i-1})q-q = \sum_{i=j}^{n}\left[\prod_{k=j}^{i} L(t_k-t_{k-1})q-\prod_{k=j}^{i-1} L(t_k-t_{k-1})q\right]$$
$$= \sum_{i=j}^{n}(t_i-t_{i-1})AL(t_i-t_{i-1})\prod_{k=j}^{i-1} L(t_k-t_{k-1})q$$
$$= \sum_{i=j}^{n}(t_i-t_{i-1})A\prod_{k=j}^{i} L(t_k-t_{k-1})q.$$

Let $p \in C$, let $c>0$, and let m be a nonnegative integer. The number-sequence $\{s_i\}_{i=0}^{m}$ is said to have property P_c provided that the following are true: (i) $s_0=0$, $s_m<\gamma_p$ (ii) $\{s_i\}_{i=0}^{m}$ is increasing, and (iii) if h is an integer in $[0, m-1]$, $s_h \leqq x \leqq s_{h+1}$, $\{t_i\}_{i=0}^{n}$ is a chain from s_h to x, and j is an integer in $[0, n]$, then

$$\left\| A\prod_{k=1}^{j} L(t_k-t_{k-1})\prod_{i=1}^{h} L(s_i-s_{i-1})p - A\prod_{k=1}^{n} L(t_k-t_{k-1})\prod_{i=1}^{h} L(s_i-s_{i-1})p\right\| \leqq c. \tag{7}$$

Lemma 1.4. *Let $p \in C$, let $c>0$, and let $\{s_i\}_{i=0}^{m}$ have property P_c. There is a number s_{m+1} such that $s_m<s_{m+1}<\gamma_p$ and $\{s_i\}_{i=0}^{m+1}$ has property P_c.*

Proof. Lemma 1.4 follows from Lemma 1.2 and the continuity of A at $\prod_{i=1}^{m} L(s_i-s_{i-1})p$.

Lemma 1.5. *Let $p \in C$, let $c>0$, and let $\{s_i\}_{i=0}^{m}$ have property P_c. Suppose that y is a number such that $s_m<y<\gamma_p$ and if s_{m+1} is a number such that $s_m<s_{m+1}<y$ then $\{s_i\}_{i=0}^{m+1}$ has property P_c. Then, if $s_{m+1}=y$, $\{s_i\}_{i=0}^{m+1}$ has property P_c.*

Proof. Let $q=\prod_{i=1}^{m} L(s_i-s_{i-1})p$, let $\{t_i\}_{i=0}^{n}$ be a chain from s_m to y, and let $d>0$. There is a positive number b such that if $u \in C$ and $\|u-\prod_{i=1}^{n} L(t_i-t_{i-1})q\|<b$ then

$$\left\|Au-A\prod_{i=1}^{n} L(t_i-t_{i-1})q\right\| < d.$$

There is a positive number r such that $t_{n-1}<r<t_n=y$ and $t_n-r<b/\|Ap\|$. By Lemmas 1.1 and 1.2

$$\left\|L(r-t_{n-1})\prod_{i=1}^{n-1} L(t_i-t_{i-1})q-\prod_{i=1}^{n} L(t_i-t_{i-1})q\right\| \leqq (t_n-r)\cdot\|Ap\| < b.$$

Then, if j is an integer in $[0, n-1]$

$$\left\|A\prod_{i=1}^{j} L(t_i-t_{i-1})q-A\prod_{i=1}^{n} L(t_i-t_{i-1})q\right\|$$
$$\leqq \left\|A\prod_{i=1}^{j} L(t_i-t_{i-1})q-AL(r-t_{n-1})\prod_{i=1}^{n-1} L(t_i-t_{i-1})q\right\|$$
$$+\left\|AL(r-t_{n-1})\prod_{i=1}^{n-1} L(t_i-t_{i-1})q-A\prod_{i=1}^{n} L(t_i-t_{i-1})q\right\|$$
$$< c+d.$$

Then, if j is an integer in $[0, n]$

$$\left\| A \prod_{i=1}^{j} L(t_i - t_{i-1})q - A \prod_{i=1}^{n} L(t_i - t_{i-1})q \right\| \leqq c$$

and so the lemma is established.

LEMMA 1.6. *Let $p \in C$, let $c > 0$, and suppose that $\{s_i\}_{i=0}^{\infty}$ is an infinite increasing number-sequence such that* $\lim \{s_i\}_{i=0}^{\infty} < \gamma_p$ *and if n is a nonnegative integer $\{s_i\}_{i=0}^{n}$ has property P_c. Then there is a positive integer m and a sequence $\{r_i\}_{i=0}^{m+1}$ such that if i is an integer in* $[0, m]$ $s_i = r_i$, $r_{m+1} = \lim \{s_i\}_{i=0}^{\infty}$, *and $\{r_i\}_{i=0}^{m+1}$ has property P_c.*

Proof. Let $q_0 = p$ and if n is a positive integer let $q_n = L(s_n - s_{n-1})q_{n-1}$. If n is a positive integer then $\|q_n - q_{n-1}\| = \|L(s_n - s_{n-1})q_{n-1} - q_{n-1}\| \leqq (s_n - s_{n-1}) \cdot \|Ap\|$. Let $s = \lim \{s_i\}_{i=0}^{\infty}$, let $q = \lim \{q_i\}_{i=0}^{\infty}$, and note that $q \in C$ since $\|q_n - p\| < s \cdot \|Ap\|$ and so $\|q - p\| < \operatorname{dist}(p, \partial C)$. There is a positive number b such that if $u \in C$ and $\|u - q\| < b$ then $\|Au - Aq\| < c/2$. Let m be a positive integer such that $\|q - q_m\| < b/2$ and $s - s_m < b/2\|Ap\|$. Let $0 < x \leqq s - s_m$, let $\{t_i\}_{i=0}^{n}$ be a chain from 0 to x, and let j be an integer in $[0, n]$. By Lemma 1.2

$$\left\| \prod_{i=1}^{j} L(t_i - t_{i-1})q_m - q_m \right\| \leqq t_j \cdot \|Ap\| < b/2$$

and so

$$\left\| A \prod_{i=1}^{j} L(t_i - t_{i-1})q_m - Aq \right\| < c/2.$$

Then, if j is an integer in $[0, n]$

$$\begin{aligned} &\left\| A \prod_{i=1}^{j} L(t_i - t_{i-1})q_m - A \prod_{i=1}^{n} L(t_i - t_{i-1})q_m \right\| \\ &\qquad \leqq \left\| A \prod_{i=1}^{j} L(t_i - t_{i-1})q_m - Aq \right\| + \left\| Aq - A \prod_{i=1}^{n} L(t_i - t_{i-1})q_m \right\| \\ &\qquad \leqq c \end{aligned}$$

and so the lemma is established.

LEMMA 1.7. *Let $p \in C$, let $c > 0$, and let $0 < x < \gamma_p$. There is a chain $\{s_i\}_{i=0}^{m}$ from 0 to x such that $\{s_i\}_{i=0}^{m}$ has property P_c.*

Proof. By Lemma 1.4 there is an infinite increasing number-sequence $\{s_i\}_{i=0}^{\infty}$ such that $\lim \{s_i\}_{i=0}^{\infty} < \gamma_p$ and if n is a nonnegative integer $\{s_i\}_{i=0}^{n}$ has property P_c. Let M denote the set of all such sequences. If $s = \{s_i\}_{i=0}^{\infty}$ is in M let $z(s)$ denote the limit of s. If each of s and t belongs to M define $s \leqq t$ only in case s is t or if n is the greatest nonnegative integer such that if i is an integer in $[0, n]$ $s_i = t_i$, then $z(s) \leqq t_{n+1}$. Then, $\leqq$ is a partial ordering of M.

Assume that there exists no member s of M such that $z(s) > x$. Let L be a linearly ordered subset of M and let y be the smallest positive number such that if s is in

L, $z(s) \leqq y$. Let $\{s_i(0)\}_{i=0}^{\infty}$, $\{s_i(1)\}_{i=0}^{\infty}, \ldots$ be an increasing sequence of points in L such that $z(s(0)), z(s(1)), \ldots$ converges to y. For each nonnegative integer i define $y_i = \sup_k s_i(k)$. Then, $y_i \leqq y_{i+1}$ and $\lim \{y_i\}_{i=0}^{\infty} = y$.

Suppose there is a positive integer n such that $y_n = y$. Then there is a least positive integer n such that $y_n = y$ and there must exist an integer k such that $s_i(k) = s_i(j)$ for each integer i in $[0, n-1]$ and $j \geqq k$. In this case $s_n(k), s_n(k+1), \ldots$ converges to y and so by Lemma 1.5 $\{s_i\}_{i=0}^{n}$, $s_i = s_i(k)$ for i in $[0, n-1]$ and $s_n = y$, has property P_c. Further, since $y < \gamma_p$, we have by Lemma 1.4 that $\{s_i\}_{i=0}^{n}$ may be extended to a member $\{s_i\}_{i=0}^{\infty}$ of M and so $\{s_i\}_{i=0}^{\infty}$ is an upper bound for L. If there is no positive integer n such that $y_n = y$ then $y_n < y$ for every n, $\{y_n\}_{n=0}^{\infty}$ is in M, $\{y_n\}_{n=0}^{\infty} \geqq s(k)$ for every k, and thus $\{y_n\}_{n=0}^{\infty}$ is an upper bound for L.

Thus, if L is a linearly ordered subset of M, then L is bounded by a member of M. By Zorn's lemma there exists $u \in M$ such that u is maximal. But then we have a contradiction since $z(u) \leqq x < \gamma_p$ and by Lemma 1.6 there exists $t \in M$ such that $u < t$. Hence, there exists $s \in M$ such that $z(s) > x$ and the lemma is proved.

LEMMA 1.8. *Let $p \in C$, let $c > 0$, and let $0 < x < \gamma_p$. There is a chain $\{s_i\}_{i=0}^{m}$ from 0 to x such that if $\{t_i\}_{i=0}^{n}$ is a refinement of $\{s_i\}_{i=0}^{m}$ then*

$$\left\| \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right\| < c. \tag{8}$$

Proof. Let $\{s_i\}_{i=0}^{m}$ be a chain from 0 to x such that $\{s_i\}_{i=0}^{m}$ has property P_c. Let $\{t_i\}_{i=0}^{n}$ be a refinement of $\{s_i\}_{i=0}^{m}$, i.e., there is an increasing sequence u such that $u_0 = 0$, $u_m = n$, and if i is an integer in $[0, m]$ $s_i = t_{u_i}$. If i is an integer in $[1, m]$ let $K_i = \prod_{j=u_{i-1}+1}^{u_i} L(t_j - t_{j-1})$, let $J_i = \prod_{j=1}^{i} L(s_j - s_{j-1})$, let $K_{m+1} = e$, and let $J_0 = e$. Then,

$$\begin{aligned}
&\left\| \prod_{i=1}^{n} L(t_i - t_{i-1})p - \prod_{i=1}^{m} L(s_i - s_{i-1})p \right\| \\
&\quad = \left\| \prod_{i=1}^{m} K_i p - J_m p \right\| \\
&\quad = \left\| \sum_{i=1}^{m} \left[\prod_{j=i}^{m} K_j J_{i-1} p - \prod_{j=i+1}^{m} K_j J_i p \right] \right\| \\
&\quad \leqq \sum_{i=1}^{m} \| K_i J_{i-1} p - J_i p \| \\
&\quad = \sum_{i=1}^{m} \| K_i J_{i-1} p - L(s_i - s_{i-1}) J_{i-1} p \| \\
&\quad \leqq \sum_{i=1}^{m} \| (e - (s_i - s_{i-1})A) K_i J_{i-1} p - J_{i-1} p \| \\
&\quad = \sum_{i=1}^{m} \left\| \left[\prod_{j=u_{i-1}+1}^{u_i} L(t_j - t_{j-1}) J_{i-1} p - J_{i-1} p \right] - (s_i - s_{i-1}) A K_i J_{i-1} p \right\|
\end{aligned}$$

$$= \sum_{i=1}^{m} \left\| \sum_{j=u_{i-1}+1}^{u_i} (t_j - t_{j-1}) \left[A \prod_{k=u_{i-1}+1}^{j} L(t_k - t_{k-1}) J_{i-1} p - A K_i J_{i-1} p \right] \right\| \quad \text{(by (6))}$$

$$\leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} (t_j - t_{j-1}) \left\| A \prod_{k=u_{i-1}+1}^{j} L(t_k - t_{k-1}) J_{i-1} p - A \prod_{j=u_{i-1}+1}^{u_i} L(t_j - t_{j-1}) J_{i-1} p \right\|$$

$$\leqq c \cdot \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} (t_j - t_{j-1})$$

$$= c \cdot x.$$

Proof of the theorem. Let $p \in C$. If $x=0$, then $\prod_0^x (e-dIA)^{-1} p = p$. If $0 < x < \gamma_p$, then $\prod_0^x (e-dIA)^{-1} p$ exists by virtue of Lemma 1.8. If $0 \leqq x < \gamma_p$ define $g_p(x) = \prod_0^x (e-dIA)^{-1} p$. By Lemma 1.2 we see that g_p is Lipschitz continuous on $[0, \gamma_p)$ with Lipschitz constant $\leqq \|Ap\|$, $g_p(x) \in C$ for $x \in [0, \gamma_p)$, and $\|Ag_p(x)\| \leqq \|Ap\|$ for $x \in [0, \gamma_p)$. For $0 \leqq x < \gamma_p$ we have that $\operatorname{dist}(p, \partial C) \leqq \operatorname{dist}(g_p(x), \partial C) + \|p - g_p(x)\| \leqq \operatorname{dist}(g_p(x), \partial C) + x\|Ap\|$. Hence,

$$\begin{aligned} \operatorname{dist}(p, \partial C)/\|Ap\| &\leqq \operatorname{dist}(g_p(x), \partial C)/\|Ap\| + x \\ &\leqq \operatorname{dist}(g_p(x), \partial C)/\|Ag_p(x)\| + x \end{aligned}$$

and so $\gamma_p - x \leqq \gamma_{g_p(x)}$. Thus, if $0 \leqq x < \gamma_p$ and $0 \leqq y < \gamma_p - x$, one sees that $g_{g_p(x)}(y) = g_p(x+y)$. To show that $g_p' = Ag_p$ let $0 \leqq x < \gamma_p$ and let $c > 0$. By Lemma 1.2 there is a positive number $z < \gamma_p - x$ such that if $0 < y < z$ and $\{s_i\}_{i=0}^m$ is a chain from 0 to y, then

$$\left\| A \prod_{i=1}^{m} L(s_i - s_{i-1}) g_p(x) - A g_p(x) \right\| < c/2.$$

Let $0 < y < z$. There is a chain $\{t_i\}_{i=0}^n$ from 0 to y such that

$$\left\| \prod_{i=1}^{n} L(t_i - t_{i-1}) g_p(x) - g_{g_p(x)}(y) \right\| < c \cdot y/2.$$

Then,

$$\begin{aligned} &\left\| \frac{1}{y} [g_p(x+y) - g_p(x)] - A g_p(x) \right\| \\ &\qquad < \frac{c}{2} + \frac{1}{y} \left\| \left(\prod_{i=1}^{n} L(t_i - t_{i-1}) g_p(x) - g_p(x) \right) - y A g_p(x) \right\| \\ &\qquad = \frac{c}{2} + \frac{1}{y} \left\| \sum_{i=1}^{n} (t_i - t_{i-1}) A \prod_{j=1}^{i} L(t_j - t_{j-1}) g_p(x) - y A g_p(x) \right\| \\ &\qquad \leqq \frac{c}{2} + \frac{1}{y} \sum_{i=1}^{n} (t_i - t_{i-1}) \left\| A \prod_{j=1}^{i} L(t_j - t_{j-1}) g_p(x) - A g_p(x) \right\| \\ &\qquad < c \end{aligned}$$

and so $g_p'^{+}(x)=Ag_p(x)$. Thus, $g_p'^{+}=Ag_p$ on $[0, \gamma_p)$ and so g_p has a continuous right derivative on $[0, \gamma_p)$. Then g_p has a continuous derivative on $[0, \gamma_p)$ and so the theorem is proved.

COROLLARY. *Let A be a mapping from the Banach space S to S such that the following are true*:

(I′) *A is dissipative on S, i.e., if $u, v \in D_A$ and $\varepsilon \geqq 0$ then* $\|(e-\varepsilon A)u-(e-\varepsilon A)v\| \geqq \|u-v\|$

(II′) $R_{(e-\varepsilon A)}=S$ *for each* $\varepsilon \geqq 0$

(III′) *A is continuous on S.*

If $p \in S$ then there is a continuously differentiable function g_p from $[0, \infty)$ *to S such that $g_p(0)=p$ and if $x \geqq 0$ $g_p'(x)=Ag_p(x)$ and* $g_p(x)=\prod_0^x (e-dIA)^{-1}p$.

Proof. The proof follows immediately from the theorem if one observes that $\alpha=+\infty$ and dist $(p, \partial S)=+\infty$.

It may be noted that a result of J. Dorroh [8] can be used to show that the solutions of $g_p'=Ag_p$, $g_p(0)=p$ in the corollary are unique. In [15] G. Minty has shown that if S is the Hilbert space then (I′) and (III′) imply (II′). More generally, it has been shown recently by T. Kato in [11] that (I′) and (III′) imply (II′) in the case that S^* is uniformly convex. If S is a general Banach space F. Browder has shown in [4] that (I′) and (III′) imply (II′) in the case that A is locally uniformly continuous.

By virtue of the corollary one may define for each $x \geqq 0$ the transformation $T(x)$ from S to S as follows: $T(x)p=g_p(x)$ for each $p \in S$. Then T is a strongly continuous semigroup of nonlinear nonexpansive transformations on S, i.e.,

(i) $T(x+y)=T(x)T(y)$ for $x, y \geqq 0$,

(ii) $T(0)=e$,

(iii) $\|T(x)p-T(x)q\| \leqq \|p-q\|$ for $x \geqq 0$ and $p, q \in S$ and

(iv) $g_p(x)=T(x)p$ is continuous for p fixed and $x \geqq 0$.

Further, A is the infinitesimal generator of T, i.e., $Ap=g_p'^{+}(0)$ for each $p \in S$. In [2], [14], [17], [18], and [19] representations are given for nonlinear nonexpansive semigroups of transformations in terms of their infinitesimal generators using product integrals.

3. **Examples.** In conclusion we give some examples. In [6] a well-known example is given by J. Dieudonné of a continuous mapping A from a Banach space S to S for which there is no solution to the equation $g'=Ag$ and $g(0)=\bar{0}$. This example is given in a Banach space which is not reflexive. Recently, J. Yorke [20] has given an example of a continuous mapping A from a Hilbert space to itself for which no solution exists to $g'=Ag$, $g(0)=\bar{0}$.

In the examples below the mapping A satisfies conditions (I′), (II′), and (III′) of the corollary.

EXAMPLE 1. Let $S=E_1$ and let A be a continuous nonincreasing function from E_1 to E_1.

EXAMPLE 2. Let $S=C_{[0,1]}$, i.e., S is the Banach space of continuous real-valued functions on [0, 1] with supremum norm. Let F be a continuous increasing function from E_1 onto E_1 such that F' is continuous and nonincreasing on E_1. Define the mapping A on $C_{[0,1]}$ as follows:

$$Af = F'[F^{-1}[f]] \quad \text{for each } f \in C_{[0,1]}.$$

The solutions g_f of the corollary are then given by $g_f(x)=F[x+F^{-1}[f]]$ for $x \geqq 0$.

In both Examples 1 and 2 A may be neither linear nor Lipschitz continuous. In both, however, A is locally uniformly continuous. In Example 3 the mapping A is not locally uniformly continuous.

EXAMPLE 3. Let $S=(c_0)$, i.e., S is the Banach space of real-number sequences $x=(x_n)$ converging to 0 with $\|x\|=\sup_n |x_n|$. If each of (a, b) and (c, d) is a point in the plane define the function $F_{[(a,b),(c,d)]}$ from $[a, c]$ to $[b, d]$ by

$$F_{[(a,b),(c,d)]}(x) = b+\left(\frac{d-b}{c-a}\right)(x-a) \quad \text{for } x \in [a, c].$$

For each positive integer n define the function A_n from E_1 to E_1 as follows:

$$\begin{aligned} A_n(x) &= 1 \quad \text{if } x < -1 \\ &= 0 \quad \text{if } x \geqq 0 \\ &= F_{[(-1/k,1/k),(-1/k+(1/n)[1/k-1/(k+1)],1/(k+1))]}(x) \quad \text{if } x \in \left[-\frac{1}{k}, -\frac{1}{k}+\frac{1}{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right) \\ &= \frac{1}{k+1} \quad \text{if } x \in \left[-\frac{1}{k}+\frac{1}{n}\left(\frac{1}{k}-\frac{1}{k+1}\right), -\frac{1}{k+1}\right) \end{aligned}$$

$$k = 1, 2, \ldots.$$

Define the mapping A from (c_0) to (c_0) by $Ax=(A_n(x_n))$ for each $x=(x_n) \in (c_0)$. One sees that A satisfies conditions (I′), (II′), and (III′), since for each positive integer n A_n is nonincreasing and continuous. Moreover, there is no neighborhood about $\bar{0}$ on which A is uniformly continuous.

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